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On Diffusive Equilibria in Generalized Kinetic Theory

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We investigate the solvability of equations

$$Q(f, f)+\epsilon^2 \Delta f=0 \qquad (\star)$$

in term of nonnegative integrable densities $f \in L^1_+(\mathbf{R}^3)$. Here, $Q(f, f)$ is a generalized collision operator. If Q is the Boltzmann operator, the only solution is 0. In contrast, we show that if Q is the pseudo-Maxwellian collision operator for granular flow, then there are non-trivial weak solutions of ($\star$).

KEY WORDS: Diffusive equilibria; kinetic granular flow.

1. INTRODUCTION

We are concerned with equations of the type

$$Q(f, f)+\epsilon^2 \Delta f=0 \qquad (1)$$

where $\epsilon>0$ is a positive constant and $Q(f, f)$ is a collision operator of Boltzmann type. Equation (1) arises in a variety of mathematical and physical contexts, such as

(i) when a diffusion term with respect to velocity is added to the full, spatially inhomogeneous Boltzmann equation in order to facilitate regularity estimates, needed for global existence and uniqueness.[(1)] Solutions of (1), if any, become the counterparts of Maxwellian equilibria for this case;

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(ii) when $Q(f, f)$ is a generalized collision operator which is such that the equation $Q(f, f) = 0$ has only the trivial solutions $f(v) = c\delta_{v_0}(v)$. This happens, for example, for rather reasonable kinetic models of traffic flow,[(2–4)] or for the generalized collision operator associated with kinetic granular flow.[(6, 7)] The latter example is the main topic of the present paper; for studies on traffic flow we refer to the above-mentioned references. The physical motivation for adding $\epsilon^2 \Delta f$ to the collision operator is, of course, that the particle system in question is immersed in a heat bath. We will refer to this effect as "diffusion" and will distinguish it from other "dissipative" effects such as convergence towards a properly normalized Maxwellian, enforced by the H-theorem, in the Boltzmann equation example.

The proper solution concept for (1) is that $f \geqslant 0$, $f \in L^1_+$, $\Delta f \in L^1$, and $\int_{\mathbf{R}^3} f(v)\, dv = C$ ($C > 0$ is a free parameter; by rescaling $\tilde{f} = \frac{f}{C}$, one obtains $\int \tilde{f} = 1$, and $\tilde{f}$ solves (1) for $\tilde{\epsilon}^2 = \epsilon^2/C$; for the rest of this paper we set $C = 1$). Additional constraints on f vary slightly from example to example.

We begin our series of results with a first, simple, slightly surprising result.

Theorem 1.1. If $Q(f, f)$ is the Boltzmann collision operator, then the only non-negative smooth solution of (1) for which the entropy production associated with $Q(f, f)$ is defined, is zero.

Proof. Suppose $f \geqslant 0$ solves (1). We multiply (1) by $\ln f$ and integrate $\int_{\mathbf{R}^3} \cdots dv$, to find

$$e(f) + \epsilon^2 \int_{\mathbf{R}^3} \Delta f \ln f = 0. \tag{2}$$

$e(f)$ is the entropy production term, and $e(f) \leqslant 0$ with equality only if f is Maxwellian. As for the second term in (2), we integrate by parts to find

$$\int_{\mathbf{R}^3} \Delta f \ln f = -\int_{\mathbf{R}^3} \frac{1}{f} |\nabla f|^2\, dv,$$

so this term is also nonpositive. It follows that (1) can only hold if $f \equiv 0$. ∎

Remark 1. If f is assumed nonnegative and such that $\int_{\mathbf{R}^3} v^2 f(v)\, dv < \infty$, then it also follows that $f \equiv 0$: multiply Eq. (1) by v^2, integrate, use energy conservation and integration by parts to find

$$0 = \int_{\mathbf{R}^3} v^2 Q(f, f)\, dv + \epsilon^2 \int_{\mathbf{R}^3} v^2 \Delta f\, dv = 0 + 6\epsilon^2 \int_{\mathbf{R}^3} f\, dv$$

so $\int_{\mathbf{R}^3} f\, dv = 0$, hence $f \equiv 0$.

These results put an end to every quest of finding diffusive equilibria for the Boltzmann equation. The diffusion term is simply stronger than the entropy-driven trend to Maxwellians, and the asymptotic effect is that particles escape with higher and higher velocity.

Remark 2. This picture must change in spatially inhomogeneous scenarios with, say, diffusive boundary conditions at the boundary $\partial\Omega$ of a bounded domain Ω; the counterpart to (1) would then be

$$v\,\nabla_x f = Q(f,f) + \epsilon^2\,\nabla_v f,$$

complemented by diffusive boundary conditions. This is a different type of problem which we do not address here.

Remark 3. For the Boltzmann–Fokker–Plank collision operator

$$Q(f,f) + \frac{1}{\tau}\,\mathrm{div}_v(vf + \theta\,\nabla_v f) = 0 \tag{3}$$

the extra drift term ensures the existence of non-trivial Maxwellians with temperature θ:

$$f(v) = \frac{\rho}{(2\pi\theta)^{3/2}}\exp\left(-\frac{v^2}{2\theta}\right).$$

This f is annihilated by both terms in (3).

In the present paper we focus on scenarios where (1) may permit non-trivial solutions, with an emphasis on the granular flow case. In Section 2 we present case studies to demonstrate the fragility of the solvability of (1) with respect to the choice of $Q(f,f)$. In Section 3 we introduce the granular flow collision operator. After restricting the generality to Maxwellian molecules we present key estimates which prove that mass and momentum are conserved by particle interactions, but energy is diminished. We then restrict our attention to the spherically symmetric case, for which the key estimate sharpens. Schauder's fixed point theorem is used to prove a general existence theorem for the spherically symmetric granular flow collision operator. In Section 4, we point out possible generalizations and present a list of open problems.

2. CASE STUDIES

We first demonstrate that the presence of a loss part to the collision term is essential for the solvability of our class of problems. To this end, we discuss two examples in one real variable, so $u = u(x)$, where $x \in \mathbf{R}$.

Example 1.

$$\epsilon^2 u_{xx} + u^2 = 0 \tag{4}$$

Theorem 2.1. There is no solution of (4) such that $u \geqslant 0$ and $\int_{-\infty}^{\infty} u(x)\, dx = 1$.

Proof. The easiest way to see this is that (4) cannot hold at inflection points $(x_0, u(x_0))$ where $u(x_0) > 0$, yet a solution satisfying $u \geqslant 0$ and $\int_{-\infty}^{\infty} u(x)\, dx < \infty$ must have such inflection points. ∎

A more systematic way of proving this is as follows (we take $\epsilon = 1$): from (4), $u_{xx} u_x + u^2 u_x = 0$, i.e.,

$$\frac{d}{dx}\left[\left(\frac{1}{2} u_x^2\right) + \frac{1}{3} u^3\right] = 0,$$

so

$$\tfrac{1}{2} u_x^2 + \tfrac{1}{3} u^3 = C. \tag{5}$$

C must be zero to satisfy the integrability condition at $\pm\infty$. Thus $u_x = \pm\sqrt{-\frac{2}{3} u^3}$, which shows that there are no solutions with $u > 0$.

Another method to prove our result follows the idea of Remark 1. We omit the details.

Example 2.

$$\epsilon^2 u_{xx} + u(u-1) = 0. \tag{6}$$

This example is a lot closer to (1). We have a "collision" operator $Q(u, u) = u^2 - u$. The solutions of (6) such that $u \geqslant 0$ and $\int_{-\infty}^{\infty} u(x)\, dx < \infty$ can be found explicitly. We first observe that it follows from (6) that if $u(\alpha) = 1$, then u has an inflection point at $x = \alpha$: $u_{xx}(\alpha) = 0$. As in Example 1 we set $\epsilon = 1$ and integrate (6) explicitly using $u(\alpha) = 1$, $u_x(\alpha) = -\beta$ as initial conditions. This yields

$$-\tfrac{1}{2}(u_x)^2 = \tfrac{1}{3} u^3 - \tfrac{1}{2} u^2 + C,$$

$$C = -\tfrac{1}{2}\beta^2 + \tfrac{1}{6}$$

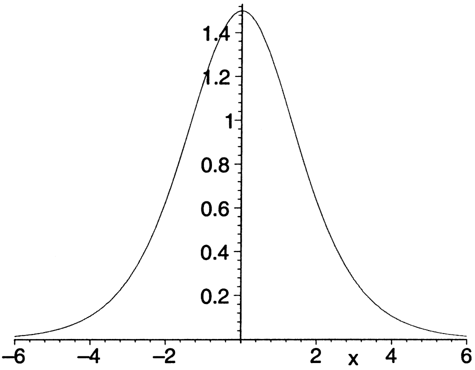

Fig. 1. Equilibrium Solution for Example 2.

or

$$u_x^2 = \beta^2 - \tfrac{1}{3} + u^2(1 - \tfrac{2}{3}u).$$

$u(x)$ and $u_x(x)$ must converge to zero as $x \longrightarrow \pm\infty$, so we see that $\beta = \frac{1}{\sqrt{3}}$. This leaves

$$u_x^2 = u^2(1 - \tfrac{2}{3}u),$$

which integrates to

$$u(x) = \frac{3}{2}\left(1 - \left(\frac{1+Ce^{-x}}{1-Ce^{-x}}\right)^2\right) \tag{7}$$

where $C = \frac{\sqrt{3}-3}{\sqrt{3}+3} e^{\alpha} < 0$. In fact, we can choose $\alpha \in \mathbf{R}$ to reach any $C \in (-\infty, 0)$. For $C = -1$, we have $u(x) = u(-x)$, and this is the unique solution which satisfies this symmetry in *addition* to being nonnegative and integrable. Equation (6) is invariant under translations $x \to x + x_0$, and other choices of C (or α) simply give translated versions of u. Specifically, to obtain the unique solution which is symmetric about $x_0 = -a$, one has to choose C such that $Ce^a = -1$, i.e., $C = -e^{-a}$.

3. GRANULAR FLOW

Let us now focus on the collision operator associated with pseudo-Maxwellian inelastic interactions in kinetic granular flows.

We use the formulation as presented in refs. 6 and 7. The collision operator $Q^e(f, f)$ is given by

$$Q^e(f, f) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \int_{\mathscr{S}^2} [f(t, \mathbf{v}_*)\, f(t, \mathbf{w}_*)\, J - f(t, \mathbf{v})\, f(t, \mathbf{w})]\, d\mathbf{n}\, d\mathbf{w}, \tag{8}$$

where $\mathbf{v}_*$ and $\mathbf{w}_*$ are the pre-collisional velocities associated to the collision mechanism,

$$\begin{cases} \mathbf{v}_* = \dfrac{1}{2}(\mathbf{v}+\mathbf{w}) - \dfrac{1-e}{4e}(\mathbf{v}-\mathbf{w}) + \dfrac{1+e}{4e}|\mathbf{v}-\mathbf{w}|\,\mathbf{n} \\[2ex] \mathbf{w}_* = \dfrac{1}{2}(\mathbf{v}+\mathbf{w}) + \dfrac{1-e}{4e}(\mathbf{v}-\mathbf{w}) - \dfrac{1+e}{4e}|\mathbf{v}-\mathbf{w}|\,\mathbf{n}, \end{cases} \tag{9}$$

$\mathbf{n} \in \mathscr{S}^2$. The number e, $0 < e \leqslant 1$, is known as the restitution coefficient. For $e = 1$ we have elastic collisions, and $Q^1(f, f) = Q(f, f)$ becomes a Boltzmann collision operator with constant collision kernel. J is the Jacobian determinant $J = |\frac{\partial(\mathbf{v}_*, \mathbf{w}_*)}{\partial(\mathbf{v}, \mathbf{w})}| = \frac{1}{e}$. Recall that the classical collision transformation (with $e = 1$) is involutive, i.e., if $\mathscr{S} : (\mathbf{v}, \mathbf{w}) \to (\mathbf{v}', \mathbf{w}')$, then $\mathscr{S}(\mathbf{v}', \mathbf{w}') = (\mathbf{v}, \mathbf{w})$. For inelastic collisions this is no longer true. While particles with velocities $\mathbf{v}_*$ and $\mathbf{w}_*$ collide to produce post-collisional velocities $\mathbf{v}$ and $\mathbf{w}$, the inverse of (9) is

$$\begin{cases} \mathbf{v}' = \dfrac{1}{2}(\mathbf{v}+\mathbf{w}) + \dfrac{1-e}{4}(\mathbf{v}-\mathbf{w}) + \dfrac{1+e}{4}|\mathbf{v}-\mathbf{w}|\,\mathbf{n} \\[2ex] \mathbf{w}' = \dfrac{1}{2}(\mathbf{v}+\mathbf{w}) - \dfrac{1-e}{4}(\mathbf{v}-\mathbf{w}) - \dfrac{1+e}{4}|\mathbf{v}-\mathbf{w}|\,\mathbf{n}, \end{cases} \tag{10}$$

i.e., $(\mathbf{v}', \mathbf{w}')$ are post-collisional velocities belonging to the pre-collisional pair $(\mathbf{v}, \mathbf{w})$.

An alternative way of writing (10) is

$$\begin{cases} \mathbf{v}' = \mathbf{v} - \alpha(\mathbf{n}'(\mathbf{v}-\mathbf{w}))\,\mathbf{n}' \\ \mathbf{w}' = \mathbf{w} + \alpha(\mathbf{n}'(\mathbf{v}-\mathbf{w}))\,\mathbf{n}', \end{cases} \tag{11}$$

where $\frac{1}{2} < \alpha \leqslant 1$. By comparing the differences $\mathbf{v}' - \mathbf{w}'$ as obtained from (10) and (11), we see that $-2\alpha(\mathbf{n}'(\mathbf{v}-\mathbf{w}))\,\mathbf{n}' = (1+e)[-\frac{1}{2}(\mathbf{v}-\mathbf{w}) + \frac{|\mathbf{v}-\mathbf{w}|}{2}\,\mathbf{n}]$; hence

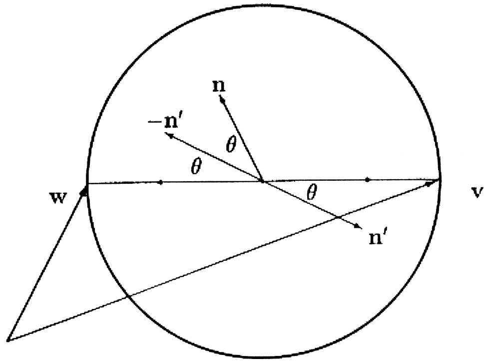

Fig. 2. Geometric meanings of $\mathbf{v}-\mathbf{w}$, $\mathbf{n}$ and $\mathbf{n}'$.

(9) and (11) define equivalent transformations if $\alpha = \frac{1}{2}(1+e)$ and if the unit vectors $\mathbf{n}'$ and $\mathbf{n}$ are related by

$$\frac{1}{2}\left(-\frac{\mathbf{v}-\mathbf{w}}{|\mathbf{v}-\mathbf{w}|}+\mathbf{n}\right) = -(\cos\theta)\,\mathbf{n}', \qquad \cos\theta = \frac{\mathbf{v}-\mathbf{w}}{|\mathbf{v}-\mathbf{w}|}\cdot\mathbf{n}'.$$

See Fig. 2 for the geometric meaning of $\mathbf{v}-\mathbf{w}$, $\mathbf{n}$ and $\mathbf{n}'$.

Note that as $-\frac{(\mathbf{v}-\mathbf{w})}{|\mathbf{v}-\mathbf{w}|}+\mathbf{n} = -2(\cos\theta)\,\mathbf{n}'$ it follows that

$$-\frac{\mathbf{v}-\mathbf{w}}{|\mathbf{v}-\mathbf{w}|}\cdot\mathbf{n}'+\mathbf{n}\cdot\mathbf{n}' = -2\cos\theta,$$

so $-\mathbf{n}\cdot\mathbf{n}' = \cos\theta$. Here, $\mathbf{n}$ and $\mathbf{n}'$ have the same azimut. θ should be thought of as the polar angle of $\mathbf{n}'$ with respect to a polar axis in direction of $\mathbf{v}-\mathbf{w}$. As $\mathbf{n}'$ sweeps $\mathscr{S}^2$, $\mathbf{n}$ will sweep $\mathscr{S}^2_+$, the upper hemisphere. The term "restitution coefficient" is explained by the identity

$$-(\mathbf{v}-\mathbf{w})\cdot\mathbf{n}' = e(\mathbf{v}_*-\mathbf{w}_*)\cdot\mathbf{n}',$$

which readily follows from

$$\begin{cases} \mathbf{v} = \mathbf{v}_* - \alpha((\mathbf{v}_*-\mathbf{w}_*))\,\mathbf{n}')\,\mathbf{n}' \\ \mathbf{w} = \mathbf{w}_* + \alpha((\mathbf{v}_*-\mathbf{w}_*))\,\mathbf{n}')\,\mathbf{n}', \end{cases}$$

and $\alpha = \frac{1}{2}(1+e)$.

We are now ready to collect crucial properties of $Q^e(f,f)$.

Lemma 3.1. For $f \in L^1_+$,

(a)

$$\int_{\mathbf{R}^3} Q^e(f,f)(\mathbf{v})\, d\mathbf{v} = 0$$

(b)

$$\int_{\mathbf{R}^3} \mathbf{v} Q^e(f,f)(\mathbf{v})\, d\mathbf{v} = 0$$

(c)

$$\int_{\mathbf{R}^3} \mathbf{v}^2 Q^e(f,f)(\mathbf{v})\, d\mathbf{v} = -\frac{1-e^2}{8} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} (\mathbf{v}-\mathbf{w})^2 f(\mathbf{v}) f(\mathbf{w})\, d\mathbf{v}\, d\mathbf{w} \quad (12)$$

Before we verify these properties, we note an immediate consequence.

Corollary 3.2. The only equilibria solutions of $Q^e(f,f)(\mathbf{v}) = 0$ are trivial, i.e., $f(\mathbf{v}) = c\delta_{\mathbf{v}_0}(\mathbf{v})$.

Indeed, for all other functions or measures the right hand side of (12) would not vanish.

Proof of Lemma 3.1. (a) and (b) are elementary; their meaning is that mass and momentum are conserved. For (c), by explicit calculations

$$\begin{aligned}
&\int_{\mathbf{R}^3} \mathbf{v}^2 Q^e(f,f)(\mathbf{v})\, d\mathbf{v} \\
&\quad = \frac{1}{8\pi} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathscr{S}^2} (\mathbf{v}^2+\mathbf{w}^2)(f(\mathbf{v}_*)\, f(\mathbf{w}_*)\, J - f(\mathbf{v})\, f(\mathbf{w}))\, d\mathbf{n}\, d\mathbf{v}\, d\mathbf{w} \\
&\quad = \frac{1}{8\pi} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathscr{S}^2} (\mathbf{v}^2+\mathbf{w}^2)\, f(\mathbf{v}_*)\, f(\mathbf{w}_*)\, J\, d\mathbf{n}\, d\mathbf{v}\, d\mathbf{w} \\
&\qquad - \frac{1}{8\pi} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathscr{S}^2} (\mathbf{v}^2+\mathbf{w}^2)\, f(\mathbf{v})\, f(\mathbf{w})\, d\mathbf{n}\, d\mathbf{v}\, d\mathbf{w} \\
&\quad := I_1 - I_2.
\end{aligned}$$

In I_1 we use the substitution $d\mathbf{v}\, d\mathbf{w} \to d\mathbf{v}_*\, d\mathbf{w}_*$. To this end, we express $\mathbf{v}^2+\mathbf{w}^2$ in terms of $\mathbf{v}_*, \mathbf{w}_*$. After some calculations using (10) (with $\mathbf{v}_*, \mathbf{w}_*$ replacing $\mathbf{v}, \mathbf{w}$ and $\mathbf{v}, \mathbf{w}$ replacing $\mathbf{v}', \mathbf{w}'$) we find

$$\mathbf{v}^2+\mathbf{w}^2 = \frac{1}{2}(\mathbf{v}_*+\mathbf{w}_*)^2 + \frac{1+e^2}{4}(\mathbf{v}_*-\mathbf{w}_*)^2 + \frac{1-e^2}{4}(\mathbf{v}_*-\mathbf{w}_*)\, |\mathbf{v}_*-\mathbf{w}_*|\, \mathbf{n}.$$

Hence

$$I_1 - I_2 = \frac{1}{8\pi} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathscr{S}^2} \Big[\frac{1}{2} (\mathbf{v}+\mathbf{w})^2 + \frac{1+e^2}{4} (\mathbf{v}-\mathbf{w})^2$$

$$+ \frac{1-e^2}{4} (\mathbf{v}-\mathbf{w})\, |\mathbf{v}-\mathbf{w}|\, \mathbf{n} - \mathbf{v}^2 - \mathbf{w}^2 \Big] f(\mathbf{v})\, f(\mathbf{w})\, d\mathbf{n}\, d\mathbf{v}\, d\mathbf{w}$$

$$= \frac{1}{8} (e^2 - 1) \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} (\mathbf{v}-\mathbf{w})^2 f(\mathbf{v})\, f(\mathbf{w})\, d\mathbf{v}\, d\mathbf{w}.$$

(here we used that $\int_{\mathscr{S}^2} (\mathbf{v}-\mathbf{w}) \cdot \mathbf{n}\, d\mathbf{n} = 0$). ∎

For the sequel we shall use the abbreviations

$$Q_+^e(f, f) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \int_{\mathscr{S}^2} f(\mathbf{v}_*)\, f(\mathbf{w}_*)\, J\, d\mathbf{n}\, d\mathbf{w},$$

$$Q_-^e(f, f) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \int_{\mathscr{S}^2} f(\mathbf{v})\, f(\mathbf{w})\, d\mathbf{n}\, d\mathbf{w}.$$

Note that if f is normalized such that $\int_{\mathbf{R}^3} f(\mathbf{v})\, d\mathbf{v} = 1$, then $Q_-^e(f, f)(\mathbf{v}) = f(\mathbf{v})$.

Corollary 3.3.

$$\int_{\mathbf{R}^3} \mathbf{v}^2 Q_+^e(f, f)(\mathbf{v})\, d\mathbf{v}$$

$$= \frac{1}{8\pi} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{\mathscr{S}^2} \Big[\frac{1}{2} (\mathbf{v}+\mathbf{w})^2 + \frac{1+e^2}{4} (\mathbf{v}-\mathbf{w})^2 \Big] f(\mathbf{v})\, f(\mathbf{w})\, d\mathbf{n}\, d\mathbf{v}\, d\mathbf{w}$$

$$= \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \Big[\frac{1}{2} (\mathbf{v}+\mathbf{w})^2 + \frac{1+e^2}{4} (\mathbf{v}-\mathbf{w})^2 \Big] f(\mathbf{v})\, f(\mathbf{w})\, d\mathbf{v}\, d\mathbf{w}$$

Proof. See the proof of Lemma 3.1. ∎

We now concentrate on the scenario where $f(\mathbf{v}) = f(|\mathbf{v}|) = f(r)$, $r = |\mathbf{v}|$, where in a mild abuse of notation we continue to use the same symbol f. For such f the identity from Corollary 3.3 becomes

$$4\pi \int_0^\infty r^4 Q_+^e(f, f)(r)\, dr = \frac{3+e^2}{8} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} (\mathbf{v}^2 + \mathbf{w}^2)\, f(|\mathbf{v}|)\, f(|\mathbf{w}|)\, d\mathbf{v}\, d\mathbf{w}. \tag{13}$$

Proof. Note that

$$\frac{1}{2}(\mathbf{v}+\mathbf{w})^2+\frac{1+e^2}{4}(\mathbf{v}-\mathbf{w})^2$$

$$=\frac{1}{2}(\mathbf{v}+\mathbf{w})^2+\frac{1+e^2}{4}(\mathbf{v}^2+\mathbf{w}^2)+\mathbf{v}\cdot\mathbf{w}(1-\frac{1}{2}(1+e^2))$$

$$=\frac{3+e^2}{8}(\mathbf{v}^2+\mathbf{w}^2)+\frac{1}{2}(1-e^2)\,\mathbf{v}\cdot\mathbf{w}.$$

If $f=f(|\mathbf{v}|)$, we see that

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\mathbf{v}\mathbf{w}\,f(\mathbf{v})\,f(\mathbf{w})\,d\mathbf{v}\,d\mathbf{w}=\int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3}|\mathbf{v}|\,|\mathbf{w}|\cos\theta f(|\mathbf{w}|)\,d\mathbf{w}\right)f(|\mathbf{v}|)\,d\mathbf{v}=0$$

(here, $\cos\theta=\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}$), and the assertion follows. ∎

In spherical coordinates the identity (13) reads

$$\int_0^\infty r^4Q_+^e(f,f)(r)\,dr=\frac{3+e^2}{4}\left(\int_0^\infty r^4f(r)\,dr\right)4\pi\int_0^\infty r^2f(r)\,dr. \tag{14}$$

In the sequel we will consider (1) under the assumption of spherical symmetry, and with the (arbitrary) normalization

$$\int_{\mathbf{R}^3}f(\mathbf{v})\,d\mathbf{v}=4\pi\int_0^\infty r^2f(r)\,dr=1. \tag{15}$$

The identity (14) then becomes

$$\int_0^\infty r^4Q_+^e(f,f)(r)\,dr=\frac{3+e^2}{4}\left(\int_0^\infty r^4f(r)\,dr\right).$$

We set $\gamma:=\frac{3+e^2}{4}$ and note that $0<\gamma<1$.

If we accept the normalization (15), Eq. (1) becomes

$$-\epsilon^2\,\Delta f+f=Q_+^e(f,f) \tag{16}$$

or, in spherical coordinates

$$-\frac{\epsilon^2}{r}\,\partial_r^2(rf)+f=Q_+^e(f,f). \tag{17}$$

After multiplication with r, we see that we are looking for a solution of

$$-\epsilon^2\,\partial_r^2(rf)+(rf)=rQ_+^e(f,f) \tag{18}$$

such that $r^2 f\in L^1_+(0,\infty)$, and normalized by (15). To solve (18), we define an operator T which will enable us to invoke a fixed point argument. Let X be the Banach space of all integrable functions on $(0,\infty)$ such that $4\pi\int_0^\infty r^2\,|f(r)|\,dr<\infty$ and $4\pi\int_0^\infty r^4\,|f(r)|\,dr<\infty$, equipped with the norm $\|f\|_2:=4\pi\int_0^\infty r^2\,(1+r^2)\,|f(r)|\,dr$ (X should be thought of as the closed subset of spherically symmetric integrable functions on $\mathbf{R}^3$ such that $\int_{\mathbf{R}^3}(1+|\mathbf{v}|^2)\,f(\mathbf{v})\,d\mathbf{v}<\infty$).

Consider the closed and convex subset D_C of X defined by

$$D_C=\left\{f\in X;\, f\geqslant 0,\, 4\pi\int_0^\infty r^2 f(r)\,dr=1,\, 4\pi\int_0^\infty r^4 f(r)\,dr\leqslant C\right\}.$$

D_C will be the domain of definition of our operator T. For $g\in D_C$, we define $f=Tg$ as the special solution of (19) which satisfies $f\geqslant 0$ and $4\pi\int_0^\infty r^2 f(r)\,dr=1$. To identify this solution, we first calculate the general solution of

$$-\epsilon^2\,\partial_r^2(rf)+(rf)=rQ_+^e(g,g)(r). \tag{19}$$

To simplify our calculations we will set $\epsilon=1$ (this is no restriction of generality). If we set $h(r)=rf(r)$, and abbreviate $\Phi=Q_+^e(g,g)(r)$, then (19) reads

$$-h''+h=r\Phi. \tag{20}$$

By variation of constants, the general solution of (20) is

$$h(r)=C_1e^{-r}+C_2e^r+\tfrac{1}{2}\int_r^\infty e^{r-t}t\Phi(t)\,dt+\tfrac{1}{2}\int_0^r e^{t-r}t\Phi(t)\,dt. \tag{21}$$

We want $rh(r)$ to be integrable; this forces the choice $C_2=0$. The choice of C_1 is then determined by the requirement that $\int_0^\infty rh(r)\,dr=\int_0^\infty r^2 g(r)\,dr$ and $h\geqslant 0$.

Lemma 3.3. If $C_1=-\frac{1}{2}\int_0^\infty te^{-t}\Phi(t)\,dt$, then $h\geqslant 0$ and $\int_0^\infty rh(r)\,dr=\int_0^\infty r^2\Phi(r)\,dr$.

Proof. First note that

$$h(r)=\tfrac{1}{2}\int_0^r e^{-r}\,(e^t-e^{-t})\,t\Phi(t)\,dt+\tfrac{1}{2}\int_r^\infty (e^r-e^{-r})\,e^{-t}t\Phi(t)\,dt, \tag{22}$$

from which $h \geqslant 0$ is obvious. The identity $\int_0^\infty rh(r)\,dr=\int_0^\infty r^2\Phi(r)\,dr$ follows by explicit integration, after interchange of the order of integration in some of the integrals. We leave the details to the reader. ∎

In a more compact form, (22) is

$$h(r)=e^{-r}\int_0^r t\sinh t\Phi(t)\,dt+\sinh r\int_r^\infty te^{-t}\Phi(t)\,dt. \tag{23}$$

Our operator T is therefore defined by the three steps $\Phi(r):=Q_+^e(g,g)(r)$, $\Phi\to h$ as in (23) and $f(r):=\frac{1}{r}h(r)$.

Lemma 3.4. If $g\in D_C$, then $f=Tg$ satisfies

$$\int_0^\infty r^2 f(r)\,dr=\int_0^\infty r^2 g(r)\,dr=\frac{1}{4\pi}, \tag{24}$$

$$\int_0^\infty r^4 f(r)\,dr=\gamma\int_0^\infty r^4 g(r)\,dr+6\int_0^\infty r^2 g(r)\,dr, \tag{25}$$

and

$$\int_0^\infty rf(r)\,dr=\int_0^\infty t(1-e^{-t})\,\Phi(t)\,dt. \tag{26}$$

Here, $\gamma=\frac{1}{4}(3+e^2)$ (see the identity after (15)).

Proof. Equation (24) is just Lemma 3.3 and Lemma 3.1a combined (note that part (a) of Lemma 3.1 reads $\int_0^\infty r^2 Q_+^e(f,f)(r)\,dr=\int_0^\infty r^2 f(r)\,dr$ $=\frac{1}{4\pi}$). For (25), we compute explicitly

$$\begin{aligned}\int_0^\infty r^4 f(r)\,dr&=\int_0^\infty r^3 h(r)\,dr\\ &=\int_0^\infty r^3 e^{-r}\int_0^r t\sinh t\Phi(t)\,dt\,dr+\int_0^\infty r^3\sinh r\int_r^\infty te^{-t}\Phi(t)\,dt\,dr\\ &=\int_0^\infty\left[\left(\int_t^\infty r^3e^{-r}\,dr\right)t\sinh t+\left(\int_0^t r^3\sinh r\,dr\right)te^{-t}\right]\Phi(t)\,dt\end{aligned} \tag{27}$$

Now,

$$\int_t^{\infty} r^3 e^{-r}\, dr = (t^3+3t^2+6t+6)\, e^{-t},$$

and

$$\int_0^t r^3 \sinh r\, dr = t^3 \cosh t - 3t^2 \sinh t + 6t \cosh t - 6 \sinh t.$$

With these integrals the bracket in (27) simplifies to t^4+6t^2. We arrive at $\int_0^\infty r^4 f(r) = \int_0^\infty (r^4+6r^2)\, \Phi(t)\, dt$, and recalling that $\Phi = Q_+^e(g, g)$ and (14), the assertion follows. Equation (26) follows by an even easier integration. ∎

The three identities (24–26) contain three key elements for a fixed point argument: (24) shows that the L_1-norm is left invariant; (25) shows that mass cannot dissipate away to infinity, and, as we will see, (26) gives us control at the origin.

We are now ready to formulate and prove the main result of this paper.

Theorem 3.4. If $C \geqslant C_0 = \frac{6}{1-\gamma}$, then the operator T has a fixed point in D_C. This fixed point is a spherically symmetric diffusive equilibrium for pseudo-Maxwellian granular flow. It is a distributional solution of (17).

Proof. The estimate

$$\int_0^\infty r^4 f(r)\, dr \leqslant \gamma \frac{C}{4\pi} + \frac{6}{4\pi} \leqslant \frac{C}{4\pi}$$

proves that T maps D_C into itself. It is elementary to prove that T is continuous with respect to the norm $\|\cdot\|_2$. To prove that $TD_C \subset D_C$ is relatively compact with respect to the L_1-norm, we first observe that

$$t(1-e^{-t}) \leqslant t^2,$$

so from (26)

$$\int_0^\infty r f(r)\, dr \leqslant \int_0^\infty t^2 Q_+^e(g, g)(t)\, dt = \frac{1}{4\pi}.$$

Therefore, for any $\epsilon > 0$

$$\int_0^{\epsilon} r^2 f(r)\, dr \leqslant \epsilon \int_0^{\infty} r f(r)\, dr \leqslant \frac{\epsilon}{4\pi},$$

i.e.,

$$\int_{|\mathbf{v}| \leqslant \epsilon} f(\mathbf{v})\, d\mathbf{v} \leqslant \epsilon. \tag{28}$$

This estimate rules out concentrations at $r = 0$. We emphasize the importance of (26) for this purpose; it is in this identity where the diffusive character of Δ comes to our help.

From $\int_0^{\infty} r^4 f(r)\, dr \leqslant \frac{C}{4\pi}$ it follows that (for $R > 1$)

$$\int_R^{\infty} r^2 f(r)\, dr \leqslant \int_R^{\infty} \frac{r^4}{R^2} f(r)\, dr \leqslant \frac{1}{R^2} \cdot \frac{C}{4\pi}. \tag{29}$$

This estimate gives us uniform control of the mass at ∞.

Finally, we see from (23) that

$$h'(r) = -e^{-r} \int_0^r t \sinh t Q_+^e(g, g)(t)\, dt + \cosh r \int_r^{\infty} t e^{-t} Q_+^e(g, g)(t)\, dt,$$

so

$$\begin{aligned}\int_0^{\infty} |h'(r)|\, dr &\leqslant \int_0^{\infty} \left[\left(\int_t^{\infty} e^{-r}\, dr \right) t \sinh t + \left(\int_0^t \cosh r\, dr \right) t e^{-t} \right] Q_+^e(g, g)(t)\, dt \\ &= \int_0^{\infty} t(1 - e^{-2t})\, Q_+^e(g, g)(t)\, dt \leqslant 2 \int_0^{\infty} t^2 Q_+^e(g, g)(t)\, dt = \frac{1}{2\pi}\end{aligned} \tag{30}$$

This estimate proves that h is of bounded variation. Hence $f(r) = \frac{h(r)}{r}$ is of uniformly bounded variation on every interval $[\epsilon, R]$, $\epsilon > 0$, $R < \infty$. In conjunction with (28) and (29), it follows that TD_C is compact in L_1.[8] The assertion of the theorem now follows from the Schauder fixed point theorem. ∎

Remark. We are stuck with a weak solution because of the possible (integrable) singularity at $r = 0$. However, we have observed that the solution f is in fact better than L_1 at $r = 0$. We believe that bootstrapping arguments and regularity theory can be involved to prove that the solution is actually $\mathbf{C}^{\infty}$. The details of this remain to be worked out—the difficulty lies in exploiting that when f is better than L_1, then $Q^e_+(f, f)$ must be better than L_1.

3.1. Generalization and Open Questions

A moment's thought shows that Theorem 3.4 generalizes readily to any collision operator which satisfies the same continuity property, mass conservation and energy dissipation as Q^e. These properties are listed in Lemmas 3.1 and 3.4.

A major gap in our results is the uniqueness question. The fixed point theorem which we used makes no assertion about uniqueness. We tried to prove uniqueness (and existence) via a contraction mapping argument but were unable to prove that T is contractive in the norm $\|\cdot\|_2$ defined earlier. It is possible that a contraction mapping argument will apply with respect to a suitably weighted L^1-norm; our efforts to find such a norm were in vain. It also is to be expected that diffusive equilibria are stable attractors for time-dependent solutions of the spatially homogeneous initial value problem.

The uniqueness for the isotropic solution (for a given normalization) is expected by analogy with the equilibrium solution for the Boltzmann equation. However, it should not be expected for fully $3D$, possibly anisotropic solutions. In fact for such solutions, existence is trivial since we have already the isotropic solution. If $E(|\mathbf{v}|)$ denotes this solution, then $E(|\mathbf{v}-\mathbf{v}_0|)$, where $\mathbf{v}_0$ is an arbitrarily given vector, is also a solution, because of the translation invariance of Eq. (16).

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